ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, No. 4, 1984

THE BANACH SPACE T AND THE FAST GROWING HIERARCHY FROM LOGIC

BY
STEVEN F. BELLENOT

ABSTRACT

Necessary and sufficient conditions for (t_k) to be equivalent to $(t_{f(k)})$ are given where (t_k) is the natural basis of the Banach space T, the so-called "Tsirelson space". The condition is in terms of the growth rate of $f(k)$ relative to the fast growing hierarchy. Roughly speaking, the ω-th level is needed to obtain nonequivalence. In particular, primitive recursive $f(k)$ yield equivalent basic sequences. The proof is obtained via some asymptotically sharp estimates on the divergence of finite-dimensional subspaces of T from l_1^d of the same dimension.

In answering a question of P. G. Casazza about the Banach space T, a collection of functions, called by logicians "the fast growing hierarchy", arises naturally. Both of these objects are of current interest in their respective fields.

The fast growing hierarchy is defined inductively as follows [8]:

$$g_0(n) = n + 1,$$

$$(*) \qquad g_{i+1}(n) = g_i^n(n),$$

where $f^n(k)$ is the n-fold composite of f, that is $f^{n+1}(k) = f(f^n(k))$; $f^1(k) = f(k)$ and $f^0(k) = k$. Hence $g_1(n) = 2n$, $g_2(n) = 2^n n$ and $g_3(n+1)$ is a "stacked tower" with n-exponents. These functions are defined for each countable ordinal α. If $\alpha = \beta + 1$, the definition $(*)$ applies. If α is a limit ordinal, then there is a "natural" sequence of ordinals $\alpha(n)$ with $\lim \alpha(n) = \alpha$ in which case $g_\alpha(n) = g_{\alpha(n)}(n)$. In particular, we have

$$g_\omega(n) = g_n(n)$$

which is as high as we will need.

Casazza's question about T goes roughly as follows. The space T has a

Received July 10, 1983 and in revised form February 20, 1984

monotone unconditional basis (t_k) which is known not to be subsymmetric. That is, there is an increasing sequence of integers $f(k)$ so that $(t_{f(k)})$ is not equivalent to (t_k). However, the existence of $f(k)$ (until now) has only been established nonconstructively via James' result [6] about l_1-isomorphs containing l_1 near-isometries. Casazza asked if for every "reasonable" $f(k)$ is (t_k) equivalent to $(t_{f(k)})$?

Theorem 1 (below) gives necessary and sufficient conditions for (t_k) to be equivalent to $(t_{f(k)})$, namely $f(k) \leqq g_l(k)$ for large k and some integer l. In particular, if $f(k)$ is primitive recursive, then the two sequences are equivalent. But the ω-th level is rather small from the logicians view, one must go to the ε_0-level (the limit of ω, ω^ω, ω^{ω^ω}, etc.) to get outside of Peano Arithmetic.

As an analysis, these growth rates are rather larger than expected. To put these results in some perspective, note by transitivity of equivalence, if $f(n)$ is strictly increasing and integer valued and if (t_k) is equivalent to $(t_{f(k)})$ then for each l, (t_k) is equivalent to $(t_{h(k)})$ for $h(k) = f^l(k)$. But in general, there is no way to diagonalize to the next higher level. For example, if (u_k) is the usual basis for J, James' quasireflexive space [7, p. 25] or for F, Figiel's reflexive space [4], then the (u_k)'s are equivalent to a shift of the basis, i.e. for $f(k) = g_0(k)$, but not for $f(k) = g_1(k) = 2k$. Indeed for F, this would imply F is isomorphic to $F \times F$ which it is not [4], and for J, we will have J isomorphic to the span of (u_{2k}) which is isomorphic to Hilbert space. For a space X with an unconditional basis, equivalence of up to $g_2(k)$ implies X is isomorphic to $X \oplus X \oplus X \oplus \cdots$. Higher $g_l(k)$'s escape the authors intuition.

For the logicians, we offer one more bit of potential interest. The result of James mentioned above has a local (i.e. finite dimensional) version. For each $\varepsilon > 0$ and $n < \infty$ there is an m so that each isomorph of l_1^m contains a $1+\varepsilon$ isomorph of l_1^n. This can be easily proved via nonstandard analysis from the infinite version, but perhaps not within Peano Arithmetic (if it can be even stated there).

The author acknowledges several enjoyable and helpful discussions with Hilbert Levitz. The author wishes also to thank P. G. Casazza for suggesting the problem and providing the author with many results about T ([1] [2] [3]).

Preliminaries

Our notation is mostly standard and generally follows that of [7]. However, we do write sums strangely. We use $\Sigma\,(a_k : m < k \leqq n)$ for $\Sigma_{k=m+1}^n a_k$ and the like. (It avoids a lot of sub- and superscript stacks.)

A basis (u_k) is said to be a *monotone unconditional basis* if

$$\left\| \sum_1^n a_k u_k \right\| \geqq \left\| \sum_1^n a_k b_k u_k \right\|$$

for each (a_k), n and any (b_k) with $\sup |b_k| \leqq 1$. A basis (u_k) is *equivalent* to a basis (v_k) if there is a constant K such that

$$K^{-1} \leqq \left\| \sum a_k u_k \right\| \Big/ \left\| \sum a_k v_k \right\| \leqq K$$

for all scalars (a_k). The smallest such K is called the *equivalence constant*. The Banach space l_1 is the collection of all scalar sequences (a_k) with norm $\|(a_k)\|_{l_1} = \Sigma |a_k| < \infty$. The usual basis has as its n-th element the sequence which is one in the n-th slot and is otherwise zero. The sup norm of a sequence is $\|(a_k)\|_0 = \sup |a_k|$.

The *Banach space T* is defined as the completion of the space of finitely non-zero scalar sequences under the norm $\|\cdot\|_T$. The norm $\|\cdot\|_T$ is inductively defined, but first some notation. Let (t_k) be the usual unit basis (as above). A sequence $(E_j)_{j=1}^k$ of finite subsets of the integers is said to be *admissible* if $k < \min E_1$ and $\max E_j < \min E_{j+1}$. We will abuse notation and also use E_j for the projection $E_j(\Sigma a_k t_k) = \Sigma (a_k t_k : k \in E_j)$. Inductively define norms on the span of the finitely non-zero sequences by $\| x \|_0$ is the sup norm and

$$\| x \|_{m+1} = \sup \left(\| x \|_0, 2^{-1} \sum_{j=1}^k \| E_j x \|_m \right)$$

where $(E_j)_{j=1}^k$ is admissible. Finally let $\| x \|_T = \lim \| x \|_m$. Note that

$$\| x \|_T = \sup \left(\| x \|_0, 2^{-1} \sum_{j=1}^k \| E_j x \|_T \right)$$

as again $(E_j)_{j=1}^k$ ranges over the admissible sequences. (This definition is given in [5] and [7, p. 95].) Because all the norms make (t_k) a monotone unconditional basis, we can assume each E_j is of the form $\{k : \min E_j \leqq k \leqq \max E_j\}$.

Results

We work backwards, pushing the technical results to the end of the paper.

THEOREM 1. *If $f(n)$ is a strictly increasing positive integer valued function then (t_k) is equivalent to $(t_{f(k)})$ if and only if there is an integer l so that $f(n) \leqq g_l(n)$ for large n.*

PROOF. A result of P. G. Casazza, W. B. Johnson and L. Tzafriri [2] (see also

[1] and [3]) states that (t_k) is equivalent to $(t_{f(k)})$ if and only if $(K_n)_{n=1}^{\infty}$ is uniformly bounded where K_n is the equivalence constant between $(t_n : f(k) < n \leqq f(k+1))$ and the unit vector basis of l_1^d $(d = f(k+1) - f(k))$. Therefore, Theorem 1 is an immediate consequence of Theorem 4 (below).

COROLLARY 2. *The subsequence* $(t_{g_\omega(k)})$ *is not equivalent to* (t_k).

COROLLARY 3. *If* $f(n)$ *is a strictly increasing positive integer valued function which is primitive recursive, then* (t_k) *is equivalent to* $(t_{f(k)})$.

PROOF. $(g_l(k))_{l=1}^{\infty}$ is co-final in the primitive recursive functions [8], that is each primitive recursive function is eventually dominated by a $g_l(k)$. □

THEOREM 4. *If* $f(n)$ *is a strictly increasing subsequence of the positive integers and* K_n *is the equivalence constant between* $(t_k : f(n) < k \leqq f(n+1))$ *and the unit vector basis of* l_1^d $(d = f(n+1) - f(n))$ *then* $(K_n)_{n=1}^{\infty}$ *is uniformly bounded is equivalent to either*:

(A) *There is an integer* l *so that* $f(n) \leqq g_l(n)$ *for large* n, *or*

(B) *There is an integer* l *so that* $f(n+1) \leqq g_l(f(n))$ *for large* n.

PROOF. Since $f(n) \geqq n$, $g_{l+1}(f(n)) \geqq g_l(g_0(f(n))) \geqq g_l(n+1) \geqq f(n+1)$, and so (A) implies (B). By Corollary 6 (below), (B) implies $K_n \leqq 2^l$ for large n. To show the converses, if (B) is false then by Corollary 6, $K_n \geqq 2^l/(l+1)$ for any l and some large n. Thus (K_n) is not uniformly bounded. Finally to see (B) implies (A), note that by induction $g_l^j(f(n)) \geqq f(n+j)$ for some large fixed n and all j. Thus for $m = f(n)$, $g_{l+1}(m+j) \geqq g_l^{m+j}(m+j) \geqq g_l^{m+j}(m) \geqq f(n+m+j) \geqq f(m+j)$. □

PROPOSITION 5. *For each* $l \geqq 0$, $n \geqq 2$ *and any* $x \in [t_k : n < k \leqq g_{l+1}(n)]$ *we have*

$$(*) \qquad \| x \|_T \geqq 2^{-l-1} \| x \|_{l_1}$$

and for some $x \in [t_k : n < k \leqq g_{l+1}(n)]$

$$(\#) \qquad \| x \|_T \leqq \| x \|_{l_1} \left(\sum_{j=1}^{l+1} 2^{-j} n^{j-l-1} \right).$$

PROOF. First we show $(*)$ is true by induction on l. If $l = 0$, then $(E_j)_{j=1}^{n}$ is admissible, where $E_j = \{n + j\}$. Thus $\| x \|_T \geqq 2^{-1} \Sigma \| E_j x \|_T = 2^{-1} \| x \|_{l_1}$. Now assume $(*)$ is true for l and let x be in the span of $\{t_k : n < k \leqq g_{l+2}(n)\}$. Now $(E_j)_{j=1}^{n}$ is admissible where $E_j = \{k : g_{l+1}^{j-1}(n) < k \leqq g_{l+1}^{j}(n)\}$ and so

$$\| x \|_T \geqq 2^{-1} \sum \| E_j x \|_T \geqq 2^{-1} \sum 2^{-l-1} \| E_j x \|_{l_1} \geqq 2^{-l-2} \| x \|_{l_1}$$

by the induction hypothesis.

To prove (#), let (a_k) be the sequence constructed in Lemma 7 below for some $l \geqq 0$. Let $x = \Sigma(a_k t_k : n < k \leqq g_l^n(n))$. Since $g_{l+1}(n) = g_l^n(n)$, we claim x is the vector in (#).

Some notation is needed. We will say a vector y is a *parent* if $\| y \|_T > \| y \|_0$, otherwise we will say y is a *leaf*. If y is a parent, its *children* are vectors $(E_j y)_{j=1}^k$ where $(E_j)_{j=1}^k$ is admissible and $\| y \|_T = 2^{-1} \Sigma_{j=1}^k \| E_j y \|_T$. (We choose one such (E_j) for each parent y.) Thus for our x as the *root* we obtain a *tree* of vectors. The *level* of a vector in this tree is the usual notion of the number of parents of parents etc. to get from that vector to the root x.

By induction, it is easy to show that

$$(\$) \qquad \| x \|_T = \sum_{j=1}^{k} 2^{-j}\Big(\sum(\| y \|_0 : y \text{ is a level } j \text{ leaf}) + 2^{-k-1} \sum(\| y \|_T : y \text{ is a level } k+1 \text{ vector})\Big)$$

if $\| x \|_T > \| x \|_0$, that is if x is a parent. If $\| x \|_T = \| x \|_0 = a_{n+1} = n^{-1}$, then since $\| x \|_{l_1} = n$, and $n \geqq 2$, the estimate (#) is true. So we assume that x is a parent. We now estimate the sums in ($), assuming $k \leqq l$.

Since $\| y \|_T \leqq \| y \|_{l_1}$ and the vectors on the same level are disjointly supported,

$$\sum(\| y \|_T : y \text{ is a level } k+1 \text{ vector}) \leqq \| x \|_{l_1} = n.$$

Next we estimate $\Sigma(\| y \|_0 : y \in L_j)$ where $L_j = \{y : y \text{ is a level } j \text{ leaf}\}$, by tracing our way back up the tree. First, $\| y \|_0 = \Sigma(a_k : m(y) < k \leqq g_0(m(y)))$ where $m(y)+1$ is the smallest integer in the support of y. For $1 \leqq i \leqq j$ let P_j^i be the set of level $j - i$ vectors which have some $y \in L_j$ as a descendent. That is P_j^1 is the set of parents of L_j and P_j^{i+1} is the set of parents of P_j^i. Thus P_j^j is either empty or $\{x\}$.

The claim is that for each $z \in P_j^i$, there is $m(z)$ so that

(1) $\{\{k : m(z) < k \leqq g_i(m(z))\} : z \in P_j^i\}$ is pairwise disjoint,

(2) $\Sigma(\| y \|_0 : y \in L_j) \leqq \Sigma(\Sigma(a_k : m(z) < k \leqq g_i(m(z))) : z \in P_j^i)$,

(3) $m(z) + 1 \geqq \min\{k : k \text{ in the support of } z\}$

The proof is by induction on i. The case for $i = 0$ is obtained by defining $P_j^0 = L_j$ and using the observation above.

Now suppose the claim is true for i, let $w \in P_j^{i+1}$ and let $C(w) = \{z \in P_j^i : z \text{ is a child of } w\}$. If we define $m(w) = \min\{m(z) : z \in C(w)\}$, then (3) is true. Furthermore, if $(z_r)_{r=1}^q$ is a list of $C(w)$ in increasing order by $m(z_r)$, then by induction hypothesis $g_i^r(m(w)) \leqq m(z_{r+1})$. Thus by condition (iii) of Lemma 7,

$$\sum(a_k : g_i^r(m(w)) < k \leqq g_i^{r+1}(m(w))) \geqq \sum(a_k : m(z_{r+1}) < k \leqq g_i(m(z_{r+1}))).$$

(Note here we are using the assumption that $j \leqq k \leqq l$ and (\$).) Hence

$$\sum (a_k : m(w) < k \leqq g_i^q(m(w))) \geqq \sum \Big(\sum (a_k : m(z) < k \leqq g_i(m(z))) : z \in C(w) \Big)$$

and since the decomposition from w to (z_r) is admissible,

$$q \leqq \min\{\min\{k : k \text{ in the support of } z_r\} - 1 : r = 1, \dots, q\} \leqq m(w).$$

Thus (2) is true since $g_i^{m(w)}(m(w)) = g_{i+1}(m(w))$. Unfortunately, (1) could be false. This technicality can be gotten around by increasing some of the $m(w)$'s. Indeed, if $(m(w_r))_{r=1}^q$ is the list in increasing order for $w_r \in P_j^{i+1}$, then keep $m(w_1)$ the same. If $m(w_2) < g_{i+1}(m(w_1))$ we can increase $m(w_2)$ to $g_{i+1}(m(w_1))$. The sum $\Sigma(a_k : m(w_1) < k \leqq g_{i+1}(m(w_1)))$ covers the sums $\Sigma(a_k : m(z) < k \leqq g_i(m(z))$; $z \in C(w_2)$; $k < g_{i+1}(m(w_1)))$ which are no longer covered by the w_2 sum. (This is the reason for the strange statement of (2).) By continuing through the list $(m(w_r))$ we can make (1) true without changing the truth of (2) or (3). This completes the claim.

Applying the claim when $i = j$ we get

$$\sum (\| y \|_0 : y \in L_j) \leqq \sum (a_k : m(x) < k \leqq g_j(m(x)))$$
$$\leqq \sum (a_k : n < k \leqq g_j(n)) = n^{l-j}$$

by conditions (iii) and (vi) of the lemma. Thus the estimate (#) is true when $k = l$, since $\|x\|_{l_1} = n$. □

Corollary 6. *If $n \geqq 2$ and $l > 1$ and if K_n is the equivalence constant between $(t_k : n < k \leqq g_l(n))$ and the usual basis of $l_1^d(d = g_l(n) - n)$, then*

(A) $2^l/(l+1) \leqq K_n \leqq 2^l$, *and*

(B) $\lim K_n = 2^l$.

Proof. To obtain (A) from (#) just replace all the n's by 2. For (B), note that only the last term does not go to zero as $n \to \infty$. □

Lemma 7. *For each $n \geqq 1$ and for each $l \geqq 0$ there is a sequence (a_k) so that*

(i) $a_k = 0$ *for* $k \leqq n$, $a_{n+1} = n^{-l}$ *and* $a_k > 0$ *for* $k > n$;

(ii) $\Sigma(a_k : m < k \leqq g_l(m)) = 1$ *for* $m \geqq n$;

(iii) *For* $0 \leqq j \leqq l$, *the function* $f(m)$ *is nonincreasing where*

$$f(m) = \sum (a_k : m < k \leqq g_j(m))), \qquad \text{for } m > n$$

(in particular that $(a_k)_{k>n}$ is nonincreasing follows from $j = 0$);

(iv) $\Sigma(a_k : A < k \leqq B) = \Sigma(a_k : g_l^j(A) < k \leqq g_l^j(B))$ *for* $j \geqq 0$, $A \geqq n$;

(v) $\Sigma(a_k : g_{l+1}(p) < k \leqq g_{l+1}(p+1)) / \Sigma(a_k : g_{l+1}(p+1) < k \leqq g_{l+1}(p+2))$ *is less than or equal to* a_{p+1}/a_{p+2} *for* $p \geqq n$; *and*

(vi) $\Sigma(a_k : n < k \leqq g_j(n)) \leqq n^{j-l}$ *for* $0 \leqq j \leqq l$.

PROOF. This is done by induction on l. If $l = 0$, we claim that $a_k = 1$ for $k > n$ satisfies the lemma. Indeed this is the only such sequence since $g_0(m) = m + 1$ and by the requirements of condition (ii). Perhaps condition (v) is the only one that is not immediate. But since $g_1(n) = 2n$, both sums are 2 and the proof is complete for $l = 0$.

Before doing the induction step, we note that (ii) and the nonincreasing nature of $(a_k)_{k>n}$ implies (iv) and (v). To start off, note that

$$\sum (a_k : m < k \leqq g_l(m)) = 1 = \Sigma(a_k : m + 1 < k \leqq g_l(m+1))$$

is true from condition (ii). Subtracting the common terms yields

$$a_{m+1} = \sum (a_k : g_l(m) < k \leqq g_l(m+1)).$$

(The above trick is used often below.)

Thus we can write:

$$\begin{aligned} a_{A+1} &= \sum (a_k : g_l(A) < k \leqq g_l(A+1)), \\ a_{A+2} &= \sum (a_k : g_l(A+1) < k \leqq g_l(A+2)), \\ &\ \ \vdots \\ a_B &= \sum (a_k : g_l(B-1) < k \leqq g_l(B)), \end{aligned}$$

and summing these implies condition (iv) when $j = 1$. Simple induction on j yields condition (iv).

For condition (v) observe from (iv) that

$$\begin{aligned} S_1 &= \sum (a_k : g_{l+1}(p) < k \leqq g_{l+1}(p+1)) \\ &= \sum (a_k : g_l^p(p) < k \leqq g^p(g_l(p+1))) \\ &= \sum (a_k : p < k \leqq g_l(p+1)) \end{aligned}$$

and similarly

$$\begin{aligned} S_2 &= \sum (a_k : g_{l+1}(p+1) < k \leqq g_{l+1}(p+2)) \\ &= \sum (a_k : p+1 < k \leqq g_l(p+2)). \end{aligned}$$

Let $x = \Sigma(a_k : p+1 < k \leqq g_l(p+1))$. Thus $S_1 = a_{p+1} + x$ and

$$S_2 = x + \Sigma(a_k : g_l(p+1) < k \leqq g_l(p+2)) = x + a_{p+2}$$

by condition (iv). Well, $a_{p+1} \geqq a_{p+2}$ implies that $f(x) = (a_{p+1} + x)/(a_{p+2} + x)$ is nonincreasing for $x \geqq 0$ and therefore condition (v) is true.

Now for the induction step, assume that (a_k) is such a sequence for l. We will obtain a nonincreasing sequence (b_k) so that $(a_k b_k)$ is the sequence for $l+1$. Observe that (b_k) nonincreasing and condition (iii) for l implies condition (iii) for $l+1$. Indeed, condition (iii) for l is equivalent to

$$0 \leqq f(m) - f(m+1) = a_{m+1} - \sum (a_k : g_j(m) < k \leqq g_j(m+1))$$

and since $b_{m+1} \geqq b_k$ for $g_j(m) < k \leqq g_j(m+1)$, it is still true for $l+1$ and $j \leqq l$. (Condition (ii) yields it for $j = l+1$.)

So let $b_k = 1$ for $k \leqq n$ and for $n < k \leqq g_{l+1}(n)$ let $b_k = 1/n$. Thus

$$\sum (a_k b_k : n < k \leqq g_{l+1}(n)) = n^{-1} \sum_{j=1}^{n} \sum (a_k b_k : g_l^{j-1}(n) < k \leqq g_l^{j}(n))$$

$$= n^{-1} n = 1.$$

This implies condition (vi), since if $j \leqq l$, then

$$\sum (a_k b_k : n < k \leqq g_j(n)) = n^{-1} \sum (a_k : n < k \leqq g_j(n))$$

$$\leqq n^{-1} n^{j-l} = n^{j-(l+1)}.$$

The remainder of the sequence (b_k) is defined inductively, a block at a time. For $g_{l+1}(n) < k \leqq g_{l+1}(n+1)$, let b_k be that constant K so that

$$a_{n+1} b_{n+1} = K\left(\sum a_k : g_{l+1}(n) < k \leqq g_{l+1}(n+1)\right).$$

Clearly this implies condition (ii) for $m = n+1$. In general, let b_q for $g_{l+1}(p) < q \leqq g_{l+1}(p+1)$ be so that

$$a_{p+1} b_{p+1} = b_q\left(\sum a_k : g_{l+1}(p) < k \leqq g_{l+1}(p+1)\right).$$

Again this choice is made to satisfy condition (ii).

Now all the other conditions will be satisfied once we show that (b_k) is a nonincreasing sequence. The first case to consider is $b_q \geqq b_{q+1}$ where $q = g_{l+1}(n)$. We have $b_q = n^{-1}$, $a_{n+1} b_{n+1} = n^{-l-1}$ and we show below that

$$\sum (a_k : g_{l+1}(n) < k \leqq g_{l+1}(n+1)) \geqq 1.$$

Thus $b_{q+1} \leqq n^{-l-1} \leqq b_q$. To see that the sum is $\geqq 1$, note that $g_{l+1}(n) = g_l^n(n)$ and

$g_{l+1}(n+1)=g_l^{n+1}(n+1)\geqq g_l^{n+1}(n)$ since $n+1\geqq n$ and all the g's are increasing functions. Thus

$$\sum(a_k : g_{l+1}(n)<k\leqq g_{l+1}(n+1))\geqq\sum(a_k : g_l^n(n)<k\leqq g_l^{n+1}(n))=1,$$

by condition (ii).

Finally, suppose $g_{l+1}(p)<q\leqq g_{l+1}(p+1)<r\leqq g_{l+1}(p+2)$. We have $a_{p+1}b_{p+1}=b_qS_1$ and $a_{p+2}b_{p+2}=b_rS_2$ where

$$S_1=\sum(a_k : g_{l+1}(p)<k\leqq g_{l+1}(p+1))$$

and

$$S_2=\sum(a_k : g_{l+1}(p+1)<k\leqq g_{l+1}(p+2)).$$

So $b_q\geqq b_r$ if and only if $a_{p+1}b_{p+1}/S_1\geqq a_{p+2}b_{p+2}/S_2$ or equivalently

$$1\geqq a_{p+2}b_{p+2}S_1/a_{p+1}b_{p+1}S_2.$$

Now by condition (v),

$$\begin{aligned}(a_{p+2}b_{p+2}/a_{p+1}b_{p+1})(S_1/S_2)&\leqq(a_{p+2}b_{p+2}/a_{p+1}b_{p+1})(a_{p+1}/a_{p+2})\\&\leqq b_{p+2}/b_{p+1}\leqq 1,\end{aligned}$$

since (b_k) is known to be nonincreasing for earlier values. This completes the lemma. □

References

1. P. G. Casazza, *Tsirelson space*, Springer lecture notes, in preparation.

2. P. G. Casazza, W. B. Johnson and L. Tzafriri, *On Tsirelson's space*, Isr. J. Math. **47** (1984), 81–98.

3. P. G. Casazza and E. Odell, *Modified Tsirelson space*, to appear.

4. T. Figiel, *An example of infinite-dimensional reflexive Banach space non-isomorphic to its Cartesian product*, Studia Math. **42** (1972), 295–305.

5. T. Figiel and W. B. Johnson, *A uniformly convex Banach space which contains no l_p*, Compos. Math. **29** (1974), 179–190.

6. R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542–550.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer, 1977.

8. C. Smoryński, "*Big*" *news from Archimedes to Friedman*, Notices Am. Math. Soc. **30** (1983), 251–256.

Mathematics and Computer Science
The Florida State University
Tallahassee, FL 32306 USA